Real Analysis HW5

P70 Q19: *Proof.* Let D be a dense set of real numbers and let f be an extended real-valued function on $\mathbb{R}$ such that $\{x : f(x) > \alpha\}$ is measurable for each $\alpha \in D$. Let $\beta \in \mathbb{R}$. For each n,there exists $\alpha_n \in D$ such that $\beta < \alpha_n < \beta + 1/n$. Now $\{x : f(x) > \beta\} = \cup\{x : f(x) \geq \beta + 1/n\} = \cup\{x : f(x) > \alpha_n\}$ so $\{x : f(x) > \beta\}$ is measurable and f is measurable. □

P71 Q24: Let $P = \{A : f^{-1}(A) \in \mathcal{M}\}$. We claim that P is a sigma algebra which contains open sets. Clearly, $\emptyset \in P$. Suppose $B \in P$, $f^{-1}(B) \in \mathcal{M}$, $f^{-1}(B^c) = [f^{-1}(B)]^c \in \mathcal{M}$. If $B_i \in P$, $f^{-1}(B_i) \in \mathcal{M}$. Then $f^{-1}(\cap B_i) = \cap_i f^{-1}(B_i) \in \mathcal{M}$. Let G be an open set. Write $G = \cup I_i$ to be disjoint union of open intervals. By definition of measurabilty, $I_i \in P$ and hence G.

P71 Q25 By continuity, $g^{-1}(a, +\infty)$ is open set. By previous Quesition, $f^{-1}g^{-1}(a, +\infty)$ is measurable.

P73 Q29 Take $f_n = \chi_{[n,\infty)}$ on the whole real line. For any A, $m(A) < 1$. We can find $x_k \to \infty$ which is outside A.

P74 Q31 By simple approximation theorem, we can find simple function $s_n \to f$. By Q3, for each s_n, we can find continuous function g_n defined on $[a, b]$, a closed set $F_n \subset [a, b]$ such that $m([a, b] \setminus F_n) < \delta/10^n$ and $s_n = g_n$ on F_n. On the other hand, we can find $F_0 \subset [a, b]$ such that s_n converges to f uniformly on F_0 and $m([a, b] \setminus F_0) < \delta/2$. Define $F = \cap_{i=0}^{\infty} F_i$ where $m([a, b] \setminus F) < \delta$. Moreover, on F, $s_n = g_n$ converges to f uniformly as $F \subset F_0$. Hence the limit function f is continuous. If the domain is $\mathbb{R}$, then we splits it into $[n, n + 1]$. On each interval, we can use the above argument to find a continuous function on $[n, n + 1]$ which approximate f. The problems arise when we glue them together which may not be continuous. However, we can modify the function around $x = n$. For instance, we give a example here. Given two continuous functions $f : [0, 1] \to \mathbb{R}$ and $g : [-1, 0] \to \mathbb{R}$. Let $\epsilon > 0$, we define

$$F(x) = \begin{cases} f(x) & \text{if } x \in [\epsilon, 1] \\ g(x) & \text{if } x \in [-1, -\epsilon] \\ ax + b & \text{if } x \in [-\epsilon, \epsilon]. \end{cases}$$

We choose a, b so that $F(x)$ is continuous function. To achieve our goal, it suffices to perform the above steps at each $x = n$ and take $\epsilon_n = \delta/100^{|n|}$.

The rest of solution can be found on the lecture note.